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MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Classical Field Theory of Matter and Electricity II. The Electromagnetic Theory of Particles

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Phil. Trans. R. Soc. Lond. A 1960 **253**, 205-226
doi: 10.1098/rsta.1960.0022

THE CLASSICAL FIELD THEORY OF MATTER AND ELECTRICITY

II. THE ELECTROMAGNETIC THEORY OF PARTICLES

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(*First submitted* 31 *May* 1954—*in revised form* 25 *February* 1959)

CONTENTS

The electromagnetic field theory developed in the previous paper is here applied to the problem of devising systems which behave as classical particles. It is found that spherically symmetrical systems can exist which, when they are stationary: (1) satisfy the static form of the extended equations at every point of space; and (2) are characterized mechanically by being everywhere in equilibrium under the sole action of the Maxwellian stress of their own field—thus they are pure electromagnetic systems subsisting free of external constraint. (3) When they are transformed so as to be in motion, the energy and momentum they possess are exactly those required for material particles by relativity theory.

A rather obvious restriction made on the generality of the conditions for particle existence brought to light the possibility of a solution denoting an 'atomic' system built up of successive shells, each of which must contain the same energy, and net charge, as the others. The reason for such a result is that, when their very great generality is restricted in the most straightforward way, the field equations reduce to the form of a wave equation. The relation of this to the wave equation of modern theory is briefly discussed.

The transformation behaviour of the field equations when a Lorentz transformation is applied to the co-ordinates is dealt with in this paper; it is found that they remain invariant in form under wider transformations of the field variables than are permitted by the classical equations. The variables may be submitted to a certain transformation without the co-ordinates being transformed at all. The physical meaning of this is investigated and an explanation of it found.

1. Introduction

The well-known classical theory of the electron, dating from the period of its discovery and now perhaps somewhat out-moded, is largely due to the great development of it by H. A. Lorentz; yet he himself drew attention to a defect in it which ever since has been a source of difficulty and doubt. He pointed out that his electron device of a charged spherical shell, contracting to a spheroid when in motion, did not fit in with the requirements of mechanical theory. A sound mechanical scheme requires the energy w and momentum $\mathbf{g}$ of a particle to be related to its velocity $\mathbf{v}$ by the equation

$$\mathrm{d}w/\mathrm{d}\mathbf{g} = \mathbf{v}, \tag{1·1}$$

for only so can the idea be developed that the increments, $\mathrm{d}w = \mathbf{F}\,\mathrm{d}\mathbf{x}$, $\mathrm{d}\mathbf{g} = \mathbf{F}\,\mathrm{d}t$ (where $\mathrm{d}\mathbf{x} = \mathbf{v}\,\mathrm{d}t$), are produced by a force $\mathbf{F}$. Yet this condition is not satisfied by the energy and momentum of the electromagnetic field surrounding the shell. As he tells us in his book *The theory of electrons* (1909), this was first pointed out by M. Abraham. Lorentz observed that the difficulty would be evaded by assuming the existence of a suitable uniform energy density inside the shell, although this could not be electromagnetic energy, there being no field there. Further H. Poincaré remarked that, if it is assumed that there is associated with the energy density an equal uniform tensional stress which acts on the charge of the shell, this will hold the latter in equilibrium against the outward pull of the electric field. One can only admire the brilliance of this pioneering work which led to the relativity formula for the mass of a moving electron; nevertheless, it soon became clear that the additional energy which had to be called in meant that the attempt to found a purely electromagnetic theory of the mass of the electron had broken down. And although this position was reached nearly 50 years ago the difficulties raised have not so far as I know been materially overcome since, in spite of numerous attempts to modify the Maxwell–Lorentz equations for the purpose.

The new terms e_t and h_t in the field equations, and the additional energy-stress to which they give rise, obtained in part I, open up, however, evident possibilities of resolving the difficulties. Their application to the problem is discussed in §§ 3, 5, 6. Section 2 contains an account of some necessary preliminary investigations of the transformation possibilities of the equations, and in § 4 some unexpected consequences of these are considered in more detail.

2. Transformation theory of field equations

When expressed in terms of ordinary space-time co-ordinates ct, x, y, z the general field equations consist of the electromagnetic identity I (4·10) subjected to the conservation conditions I (4·13). In order that these should constitute a valid set of equations it is necessary that each should remain invariant in form when the co-ordinates are subjected to a Lorentz transformation. The following investigation not only shows that this is so, but also reveals an invariance in form to another type of transformation which is of some importance in the general theory of the equations.

(a) *Invariance of form of electromagnetic identity*

Discussion of the transformations is much simplified by dealing with the representation I (4·12a, b, c) of the identity in Euclidean space, as the transformations can then be treated formally as real rotations. The general problem presented is: in the identity,

$$\bar{\partial}\partial\phi = q, \quad z = \bar{\partial}\mathrm{R}\phi, \quad \bar{\partial}\mathrm{R}^{-1}z \equiv q,$$

what individual rotations can be given to the three vectors ∂, z, q without altering the form of the equations? It can be solved by making use of a result (B (2·14)) obtained in the earlier papers that, if any 4^2 matrix A undergoes the rotation

$$A \to \bar{c}\mathrm{R}\,\bar{d}\mathrm{S}\,.\,A\,.\,\bar{e}\mathrm{R}\,\bar{f}\mathrm{S}, \tag{2·1 a}$$

then its e-transform ${}^eA$ undergoes the rotation

$${}^eA \to \bar{c}\mathrm{R}\,\bar{e}\mathrm{S}^{-1}\,.\,{}^eA\,.\,\bar{d}\mathrm{R}^{-1}\bar{f}\mathrm{S} \tag{2·1 b}$$

(c, d, e, f are here arbitrary unit vectors). To apply this, let $[\partial]$ denote the 4^2 matrix which has $|(\partial_1\,..\,\partial_4)$ for its first column, and all the other terms zero, then $\bar{\partial}\mathrm{R} = 2\,{}^e[\partial]$ (B (2·6)). Consequently, if the vector ∂ undergoes a rotation to

$$\partial' = \bar{c}\mathrm{R}\,\bar{d}\mathrm{S}\partial, \quad \text{i.e.} \quad [\partial]' = \bar{c}\mathrm{R}\,\bar{d}\mathrm{S}\,.\,[\partial]\,.\,\mathbf{1}, \tag{2·2 a}$$

then the vector system $\bar{\partial}\mathrm{R}$ undergoes a rotation to

$$\bar{\partial}'\mathrm{R} = \bar{c}\mathrm{R}\,.\,\bar{\partial}\mathrm{R}\,.\,\bar{d}\mathrm{R}^{-1}. \tag{2·2 b}$$

Hence we get the mathematical identity

$$\overline{(\bar{c}\mathrm{R}\,\bar{d}\mathrm{S}\partial)}\,\mathrm{R} \equiv \bar{c}\mathrm{R}\,.\,\bar{\partial}\mathrm{R}\,.\,\bar{d}\mathrm{R}^{-1} \tag{2·2 c}$$

(the long bar denotes that the resultant vector below it is written in row form). The equivalence of these expressions can be applied to the equation $z = \bar{\partial}\mathrm{R}\phi$, for, multiplying both sides of it by $\bar{c}\mathrm{R}\,\bar{f}\mathrm{S}$, we get

$$\bar{c}\mathrm{R}\,\bar{f}\mathrm{S}z = \bar{c}\mathrm{R}\,\bar{f}\mathrm{S}\,.\,\bar{\partial}\mathrm{R}\,.\,\phi = \bar{c}\mathrm{R}\,.\,\bar{\partial}\mathrm{R}\,.\,\bar{f}\mathrm{S}\phi = \bar{c}\mathrm{R}\,.\,\bar{\partial}\mathrm{R}\,.\,\bar{d}\mathrm{R}^{-1}\,\bar{d}\mathrm{R}\,\bar{f}\mathrm{S}\phi,$$

giving

$$\bar{c}\mathrm{R}\,\bar{f}\mathrm{S}z = \overline{(\bar{c}\mathrm{R}\,\bar{d}\mathrm{S}\partial)}\,\mathrm{R}\,.\,\bar{d}\mathrm{R}\,\bar{f}\mathrm{S}\phi. \tag{2·3}$$

Equation (2·3) gives the most general rotations which can be applied to the three vectors z, ∂, ϕ consistent with maintaining the relation $z' = \bar{\partial}'\mathrm{R}\phi'$ between their rotated values. Each in general must rotate in a different, though not completely independent, way. If we try to make them all undergo the same rotation, or (what is equivalent) rotate the axes and expect the vectors to remain fixed in the 4-space, the common rotation cannot be more extensive than $\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}$. This is a rotation confined to xyz-space; for all such the vectors do behave as standard vectors in the tensor calculus, but this is no longer the case when the rotations extend into the time dimension.

Finally, since $\bar{\partial}\partial$ is invariant and so q must rotate in the same way as ϕ, the rotations of (2·3) hold in $\bar{\partial}\mathrm{R}^{-1}z - q = 0$. It follows that the space-time equivalents of the identity, (I (4·10), (4·13)), continue to hold with the space-time equivalents of the transformed variables

$$\left.\begin{aligned} \partial' = \bar{c}\mathrm{R}\,\bar{d}\mathrm{S}\partial, \quad z' = e' + \mathrm{i}h' = \bar{c}\mathrm{R}\,\bar{f}\mathrm{S}(e + \mathrm{i}h), \\ q' = r' + \mathrm{i}s' = \bar{d}\mathrm{R}\,\bar{f}\mathrm{S}(r + \mathrm{i}s), \quad \phi' = \psi' + \mathrm{i}\chi' = \bar{d}\mathrm{R}\,\bar{f}\mathrm{S}(\psi + \mathrm{i}\chi). \end{aligned}\right\} \tag{2·4}$$

Classical transformations a special case

The transformations (2·4) are wider than those the standard electromagnetic theory allows for $\mathbf{e}$, $\mathbf{h}$, j_t and $\mathbf{j}$. Let us see first how they include these. We may dismiss the case of the general rotation $\bar{c}\mathrm{R}\,\bar{d}\mathrm{S}$ of ∂ as it merely leads to the complication of complex co-ordinates, and confine attention to the inverse Lorentz transformation I (2·11), which changes the co-ordinate vectors x and x into respectively

$$x' = c\mathrm{R}\,c\mathrm{S}^{-1}x, \quad \mathrm{x}' = L^{-1}\mathrm{x}. \tag{2·5}$$

The changed values of the corresponding covariant ∂ and D are obtained by applying the standard tensor rules given in matrix form in B (1·2), and are

$$\partial' = \overline{(\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}^{-1})}^{-1}\,\partial = \bar{c}\mathrm{R}\,\bar{c}\mathrm{S}^{-1}\partial, \quad \mathrm{D}' = \overline{(L^{-1})}^{-1}\,D = L\mathrm{D}, \tag{2·6}$$

since $\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}^{-1}$ is orthogonal and L symmetric. (To introduce $\bar{c}\mathrm{S}^{-1}$ into (2·4) put $d = \eta c$, noting that $\bar{\eta}\bar{c}\mathrm{S} = \bar{c}\mathrm{S}^{-1}$.) The most general rotational transform of

$$\bar{\partial}\mathrm{R}^{-1}(e+\mathrm{i}h)-(r+\mathrm{i}s) = 0,$$

consistent with the inverse Lorentz transformation of the co-ordinates is consequently

$$\overline{(\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}^{-1}\partial)}\,\mathrm{R}^{-1}.\bar{c}\mathrm{R}\bar{f}\mathrm{S}(e+\mathrm{i}h)-\bar{c}\mathrm{R}^{-1}\bar{f}S(r+\mathrm{i}s) = 0. \tag{2·7}$$

Superposed on the determinate rotations, $\bar{c}\mathrm{R}$ and $\bar{c}\mathrm{R}^{-1}$, respectively, which $e+\mathrm{i}h$ and $r+\mathrm{i}s$ must undergo in consequence of that of ∂, a common rotation $\bar{f}\mathrm{S}$, in which f is arbitrary, can be applied to both.

The classical transformations are arrived at by making the special choice for f if $f = c$, then

$$e'+\mathrm{i}h' = \bar{c}\mathrm{R}\,\bar{c}\mathrm{S}(e+\mathrm{i}h), \quad r'+\mathrm{i}s' = \bar{c}\mathrm{R}^{-1}\,\bar{c}\mathrm{S}(r+\mathrm{i}s). \tag{2·8 a, b}$$

As regards the first, it is easy to verify by using I (2·9), (2·10), (2·13) and the conversion formula I (2·2a), that it gives the same results for $e_x\,..\,e_z$, $h_x\,..\,h_z$ as are obtained in standard electromagnetic theory. The two transformations, nevertheless, are quite different in form, so it is interesting to trace the reason why in spite of this they give the same result. The relation between the complex vector $e+\mathrm{i}h$ and the 2nd rank antisymmetric tensor $\mathrm{F}^{\mu\nu}$ of the standard equations is expressed in a simple way in E-number theory. The matrix F which denotes $[\mathrm{F}^{\mu\nu}]$ in Euclidean 4-space is obtained from it by evaluating $\zeta[\mathrm{F}^{\mu\nu}]\,\bar{\zeta}$, and is given in terms of $e_x\,..\,h_z$ in (2·9a). On the right the e-transform of F determined as described in A § 3 (h), is also given

$$F = \begin{bmatrix} . & \mathrm{i}e_x & \mathrm{i}e_y & \mathrm{i}e_z \\ -\mathrm{i}e_x & . & -h_z & h_y \\ -\mathrm{i}e_y & h_z & . & -h_x \\ -\mathrm{i}e_z & -h_y & h_x & . \end{bmatrix}, \quad {}^eF = \begin{bmatrix} . & \mathrm{i}e_x+h_x & \mathrm{i}e_y+h_y & \mathrm{i}e_z+h_z \\ -\mathrm{i}e_x+h_x & . & . & . \\ -\mathrm{i}e_y+h_y & . & . & . \\ -\mathrm{i}e_z+h_z & . & . & . \end{bmatrix}. \tag{2·9 a, b}$$

On conversion by (2·18), the first column of ${}^eF$ becomes the 234 terms of $-(e+\mathrm{i}h)$, and the first row those of $-(e-\mathrm{i}h)$. Thus the relation is that a 4^2 *matrix which has the space components of* $-(e+\mathrm{i}h)$ *as the space terms of its first column, and their conjugates as those of its first row, with all its other terms zero, and the matrix F which expresses* $[\mathrm{F}^{\mu\nu}]$ *in Euclidean* 4-*space, are e-transforms of each other*; i.e. with $e_1, h_1 = 0$,

$${}^eF = -\{[e+\mathrm{i}h]+[e-\mathrm{i}h]\}, \quad {}^e({}^eF) = F. \tag{2·9 c}$$

The relation enables us to determine the transformation of F which corresponds to (2·8a); for, since $\overline{e'-\mathrm{i}h'} = \overline{e-\mathrm{i}h}\,\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}$, as is easily seen, we have

$$({}^eF)' = \bar{c}R\,\bar{c}\mathrm{S}\,.\,{}^eF.\,\bar{c}\mathrm{R}\,\bar{c}\mathrm{S}, \tag{2·10 a}$$

and consequently by (2·1)

$$F' = \bar{c}\mathrm{R}\,\bar{c}\mathrm{S}^{-1}.\,F.\,\bar{c}\mathrm{R}^{-1}\,c\mathrm{S}. \tag{2·10 b}$$

This denotes an inverse Lorentz rotation of both columns and rows of F, and when converted into space-time terms reduces to the classic form of the transformation.

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The relation between the complex 3-vector $\mathbf{h}+i\mathbf{e}$ and $F^{\mu\nu}$ equivalent to (2·9) was derived by Laporte & Uhlenbeck (1931) in terms of the spinor notation of van der Waerden's theory. They have also by this means derived the classic electromagnetic equations as complex equations in $\mathbf{h}+i\mathbf{e}$. So far as I can see, van der Waerden's spinor theory does not include in its scope the additional terms e_t and h_t, which are an essential part of a general 4-vector field, nor the more extensive transformation (2·7) which follows from the application of E-number theory—but I do not know about possibilities of later extensions of the theory to 4-suffixed spinors. In any case it is not possible to generalize F as a single matrix to include the case of finite e_t and h_t. For the general field the complex-vector description seems much the simplest.

As regards $r+is$ in (2·8) it will be observed that it undergoes a direct Lorentz rotation, so that the space-time r transforms into $r' = Lr$, not like x but like a covariant vector. It is not, however, r but the charge and current vector j with whose behaviour the standard theory is concerned. Since e_t is now invariant, we have by I (4·10*a*)

$$j' = \eta(r' - D'e_t) = \eta L(r - De_t) = \eta L . \eta j = L^{-1} j \qquad (2{\cdot}11)$$

which is the classical formula.†

Nevertheless, the classical rotation transformations of e, h, and j form only a particular choice out of an infinite number of them that leave invariant the form of the equation. The f of the rotation $\bar{f}S$ in (2·4) is completely arbitrary so long as it is independent of x, and *this* S-*rotation may be applied at will to the complex vectors of the identity without changing the coordinate system in any way*. For if each of the three equations of the identity is pre-multiplied through by $\bar{f}S$, this commutes with the differentiator, so that it may be taken to the right of it and applied to the vectors.

(*b*) *Invariance of form of conservation restriction*

The following proof that equation I (4·13*b*), which can be written compactly as

$$\tfrac{1}{2}\{(e+ih)\, S_k(r-is) + (r+is)\, S_k(e-ih)\} = 0 \qquad (2{\cdot}12a)$$

and which expresses these conditions, retains its form when $e+ih$ and $r+is$ undergo the general rotations (2·8) is based on the rules for the multiplication of members of a quaternionic set (in this case S) derived in paper *A*. With these values (2·8) the left-hand side of I (4·13*b*) is readily seen to be

$$\tfrac{1}{2}\{(\bar{e}+i\bar{h})\bar{f}S^{-1}\bar{c}R^{-1}.S_k.\overline{c^*}R^{-1}\overline{f^*}S(r-is) + (\bar{r}+i\bar{s})\bar{f}S^{-1}\bar{c}R.S_k.\overline{c^*}R\overline{f^*}S(e-ih)\}. \qquad (2{\cdot}12b)$$

Since c is a 'simple-term' vector $(|(c_t, ic_x, ic_y, ic_z))$, whence $\overline{c^*}R^{-1} = \bar{c}R$, the R-factors cancel, as they commute with the S, so it is clear that as far as the R-part of the rotations is concerned (2·12) is invariant. But neither of these simplifications applies to $\bar{f}S$, for f might be chosen if we wish to be a fully complex vector. We have, however,

$$\bar{f}S^{-1}S_k\overline{f^*}S = f_\alpha f^*_\beta S^{-1}_\alpha S_k S_\beta = f_\alpha f^*_\beta S^{-1}_\alpha R_{kj\beta} S_j,$$

† The last conversion in (2·11) illustrates a property of the Lorentz matrix due to the fact that $\eta\,.\,.\,\eta$ converts R_α into S_α and vice versa, thus, by I (2·12) and A (2·14),

$$\eta L \eta = \zeta^{-1}\eta\bar{c}R^{-1}\eta . \eta\bar{c}S\eta\zeta = \zeta^{-1}\bar{c}S^{-1}cR\zeta = L^{-1}.$$

on evaluating the product $S_k S_\beta$ by A (2·23c), and on further evaluating $S_\alpha^{-1} S_j$ by (b) of the same set of equations,

$$\bar{f}S^{-1}S_k\overline{f^*}S = f_\alpha f_\beta^* R_{\alpha jl} R_{kj\beta} S_l = (\bar{f}R^{-1}.R_k f^*)_l S_l = a_l^{(k)} S_l, \quad \text{say}, \tag{2·13}$$

since the content of the brackets for fixed k denotes a vector. Consequently (2·12) takes the form

$$\tfrac{1}{2}a_l^{(k)}\{(\bar{e}+i\bar{h})\, S_l(r-is)+(\bar{r}+i\bar{s})\, S_l(e-ih)\}, \tag{2·14}$$

and so the transformed expression of I (4·13b) for a given k is simply the sum of the four original expressions each multiplied by an appropriate factor, and since these vanish, it remains zero.

The extended transformation of $e+ih$ also leaves *the general conservation equation* $\partial_i Z_{ik} = 0$ (I (4·14)) *invariant in form*. With the transformations of (2·8)

$$(e+ih)\,(\bar{e}-i\bar{h}) \to \bar{c}R\bar{f}S.(e+ih)\,(\bar{e}-i\bar{h}).\bar{c}R\overline{f^*}S^{-1} \tag{2·15a}$$

and the e-transform Z of this by (2·1) consequently undergoes the rotation

$$Z \to \bar{c}R\,\bar{c}S^{-1}.Z.\bar{f}R^{-1}\overline{f^*}S^{-1}. \tag{2·15b}$$

(If as in (2·8) we put for the arbitrary f the simple-form value c, the post-factor of Z becomes $\bar{c}R^{-1}\bar{c}S$. This makes the rotation which Z undergoes in the fourfold reduce to the (inverse) Lorentz rotation of a standard 2nd-rank tensor, and brings Z (apart from the e_t, eh_t which it contains) into full accordance with the classical theory.)

In general, in the transformed system we shall have

$$\begin{aligned}\partial'_i Z'_{ik} &= (\bar{c}R\,\bar{c}S^{-1}\partial)_i\,(\bar{c}R\,\bar{c}S^{-1}\,Z\bar{f}R^{-1}\overline{f^*}S^{-1})_{ik}\\ &= (\bar{c}R\,\bar{c}S^{-1})_{ij}\,(\bar{c}R\,\bar{c}S^{-1})_{il}\,(\bar{f}R^{-1}\overline{f^*}S^{-1})_{mk}\,\partial_j Z_{lm}\\ &= (\bar{f}R^{-1}\overline{f^*}S^{-1})_{mk}\,\partial_j Z_{jm},\end{aligned} \tag{2·16}$$

since the product of the first two factors is the jlth term of the unit matrix, which substitutes j for l in Z_{lm}. It is clear that when $\partial_i Z_{ik} = 0$ the transformed equation will also be zero, even when $e+ih$ has undergone the arbitrary rotation $\bar{f}S$, as the effect of this is merely to form a sum of a number of zero terms.

3. The construction of charged particles

(a) *Lorentz electron*

Consider a static field consisting of a scalar h_t of uniform magnitude

$$h_t = q/(4\pi a^2) \quad (q \text{ const.}) \tag{3·1}$$

inside a stationary sphere of radius a centred at the origin, and outside it a radial electric force $\mathbf{e}$ of magnitude

$$e_r = q/(4\pi r^2) \tag{3·2}$$

at distance r from the centre. These values satisfy the electromagnetic equations I (4·11) with $r_t \ldots s_z$, alternatively $j_t \ldots k_z$, zero, everywhere inside and outside the sphere. They are, however, discontinuous at $r = a$, where we must suppose that h_t falls to zero, and e_r rises from zero, though an infinitely thin shell at the surface of the sphere. If δa is the thickness of this shell we shall have in it

$$\operatorname{div}\mathbf{e} = e_r/\delta a = j_t = r_t, \quad \text{since} \quad e_t = 0. \tag{3·3a}$$

There is thus an infinite electric charge density in the shell, and q denotes the total charge. Also, in the shell curl $\mathbf{e} = 0$, on assuming $\mathbf{e}$ is symmetrical, and so for a static solution of (4·11) we must have

$$k_t = 0, \quad \mathbf{k} = 0, \tag{3·3b}$$

where $\mathbf{k}$ is the three-vector denoting $(1/c)$ times the magnetic current density. In the standard theory no value other than zero is ever contemplated for k_t and $\mathbf{k}$, but here, in view of I (4·11) they must be made zero by assuming for s_t and $\mathbf{s}$ the values

$$s_t = 0, \quad \mathbf{s} = \operatorname{grad} h_t = -\frac{q}{4\pi a^2 \delta a}\mathbf{r}_1 \quad (\mathbf{r}_1 = \text{unit radial vector}). \tag{3·3c}$$

(1) With these values we shall have a static solution of the extended electromagnetic equations valid over all space including the charged shell at the surface of the sphere. It differs from the standard theory not only by the presence of h_t in the sphere, but also of $\mathbf{s}$ in the shell. The magnetic current is zero, not absolutely, but as the result of a balance of two opposite quasi-currents, whose densities are $\mathbf{s}$ and grad h_t.

(2) Considered as a field of mechanical stress the system described is everywhere in statical equilibrium. Inside the sphere the general stress system Z (I (3·15)) reduces to a uniform tension $\frac{1}{2}h_t^2$ (obviously in self equilibrium)—outside to that of the electrostatic field (3·2), a radial tension of $\frac{1}{2}e_r^2$ along with an equal circumferential pressure, well known to be in equilibrium. Within the thickness of the charged shell the two systems balance each other. One way of verifying these statements is to observe that the conservation conditions I (4·13b), which in a static field, reduce to the conditions for the equilibrium of the stress, are here satisfied. For example, in the shell, on omitting the terms which are evidently zero, I (4·13b) reduces to the three-vector equation

$$-h_t\mathbf{s} - r_t\mathbf{e} + [\mathbf{es}] = 0 \quad ([\,.\,.\,] = \text{vector product}), \tag{3·4}$$

and this is identically satisfied by the values in (3·1) to (3·3).

(3) A third significant point is that when this system is regarded from the standpoint of an observer moving with velocity $-v$ along the axis of x, to whom therefore its centre appears to be in motion with velocity $+v$, and the field variables are given the corresponding standard transformations, the total energy and the x-momentum which the system now possesses are related to the velocity precisely as are those of a particle on relativity theory. As has been pointed out in § 1 this is what was proved by Lorentz, except that, as energy inside the sphere could not be accounted for by classical electromagnetic theory, he had to confine himself to pointing out that such energy would be necessary if this result were to be produced. Although Lorentz's conclusion is well known, it will be useful to give a brief proof for the extended system here dealt with of the mechanical property (3), as this will help to make clearer some of the developments which follow.

The energy and momentum of the system when in motion are calculated as consequences of submitting the co-ordinates to an inverse Lorentz transformation along OX. This gives the system a velocity $+v$ along OX.

Let the centre C of the static system coincide with the origin O, and consider the field at a point P (where $CP = r$, angle $PCX = \theta$, so that $x = r\cos\theta$, $(y^2+z^2)^{\frac{1}{2}} = p = r\sin\theta$), when the co-ordinates, after transformation to the accented system with origin O', have become

$$x' = \beta(x+vt), \quad p' = p \quad \text{at} \quad t' = \beta(t+vx/c^2), \tag{3·5}$$

where $\beta = 1/(1-v^2/c^2)^{\frac{1}{2}}$. It has been seen in § 2 that the classical transformations of the field variables are valid for the extended equations, and they are there supplemented by h_t = invariant. We readily get for the transformed fields consequent on the inverse Lorentz transformation

$$h'_t = h_t, \quad e'_x = e_x, \quad e'_p = \beta e_p, \quad h'_\phi = \alpha\beta e_p, \tag{3·6}$$

where $\alpha = v/c$, and the suffix ϕ to a positive term denotes a field circling right-handedly about $+OX$. As these components are affected in different ways by the transformation, the total energy due to each of them has to be calculated separately. Take as an example the energy of the e_p-field.

In the original static system the

$$e_p\text{-energy} = \int_0^\pi \int_a^\infty \frac{1}{2}\left(\frac{q\sin\theta}{4\pi r^2}\right)^2 2\pi r\sin\theta\, r\,\mathrm{d}r\,\mathrm{d}\theta \tag{3·7 a}$$

which is easily integrated to $q^2/12\pi a$. But in the transformed system we must write, making the calculation for a fixed time t',

$$e'_p\text{-energy} = \int_0^\pi \int_{a'}^\infty \frac{1}{2}\left(\frac{\beta q\sin\theta}{4\pi r^2}\right)^2 2\pi r'\sin\theta' r'\,\mathrm{d}r'\,\mathrm{d}\theta', \tag{3·7 b}$$

where r' and θ' are the values which $C'P'$ and $P'C'X'$ must now possess, and a' is the radius in the direction θ' of the spheroid which the original sphere has now become.

Equation (3·7 b) cannot be integrated without formulae for converting r and θ into r' and θ', and these may be obtained as follows: since C' is moving with velocity v with respect to the new origin O', which coincides with O at $t' = 0$, we have at time t'

$$r' = \{(x'-vt')^2+p'^2\}^{\frac{1}{2}}, \quad \cos\theta' = (x'-vt')/r'. \tag{3·8 a}$$

Also, since from (3·5) $x = \beta(x'-vt')$, the r, θ, of P in the original system to which P' corresponds, are expressible as

$$r = \{\beta^2(x'-vt')^2+p'^2\}^{\frac{1}{2}}, \quad \cos\theta = \{\beta(x'-vt')\}/r. \tag{3·8 b}$$

From these equations it is easy to deduce the conversion formulae

$$\left.\begin{aligned} r' &= Fr, \quad \sin\theta' = \sin\theta/F, \quad \cos\theta' = \cos\theta/\beta F, \quad \mathrm{d}\theta' = \mathrm{d}\theta/\beta F^2,\\ r &= \beta F' r', \quad \sin\theta = \sin\theta'/\beta F', \quad \cos\theta = \cos\theta'/F', \quad \mathrm{d}\theta = \mathrm{d}\theta'/\beta F'^2,\\ \text{where} \quad F &= (1-\alpha^2\cos^2\theta)^{\frac{1}{2}}, \quad F' = (1-\alpha^2\sin^2\theta')^{\frac{1}{2}} \quad FF' = 1/\beta. \end{aligned}\right\} \tag{3·9}$$

Using one or the other of these we can convert (3·7 b) wholly into terms of r' and θ' (the logical way, but it leads to a complicated expression), or alternatively of r and θ, using these simply as a substitution device for effecting the integration. The latter way gives

$$e'_p\text{-energy} = \int_0^\pi \int_a^\infty \beta^2 \frac{q^2\sin^2\theta}{16\pi r^4} r^2\,\mathrm{d}r^2 F^3 \frac{\sin\theta}{F}\frac{\mathrm{d}\theta}{\beta F^2} = \beta\times(e_p\text{-energy}) \quad \text{by (3·7 a)}. \tag{3·10}$$

The total energies, now denoted by w_0 and w, before and after transformation, separated into terms showing in order the h_t, e_x, e_p, h_ϕ-constituents, are given below

$$\left.\begin{aligned} w_0 &= \frac{q^2}{24\pi a}(1+1+2+0) = \frac{q^2}{6\pi a},\\ w &= \frac{q^2}{24\pi a}\left(\frac{1}{\beta}+\frac{1}{\beta}+2\beta+2\beta\alpha^2\right) = \beta\frac{q^2}{6\pi a}. \end{aligned}\right\} \tag{3·11}$$

Thus

$$w = w_0/(1-v^2/c^2)^{\frac{1}{2}}$$

exactly as in the relativity theory of a particle.

The momentum is all outside the spheroid with the density

$$\mathbf{G}' = \frac{1}{c}[\mathbf{e}'\mathbf{h}'].$$

The total x-momentum is consequently

$$g'_x = \int_0^\pi \int_{a'}^\infty \frac{1}{c} e'_p h'_\phi 2\pi r'^2 \sin\theta' \,\mathrm{d}r' \,\mathrm{d}\theta'$$
$$= \frac{1}{c}\frac{q^2}{6\pi a}\frac{v/c}{(1-v^2/c^2)^{\frac{1}{2}}}. \tag{3·12}$$

The momentum perpendicular to x balances itself out, so g'_x is all that can be observed, and, writing it g we have

$$g = \frac{v}{c^2}w, \tag{3·13}$$

just as with a mechanical particle. These results of course satisfy the essential condition (1·1).

(b) *Any spherically symmetrical charge*

The properties marked (1) to (3) in the last section are not confined to the Lorentz electron, in which the charge is restricted to a thin shell. It can be shown that on the extended equations a system containing *any* spherically symmetrical distribution of electric charge can be constructed so as to possess them.

On assuming that the only field variables are h_t, e_x, e_y, e_z, and that there is no momentum or flux of energy, the energy-stress matrix Z (I (3·15)) takes the form

$$\begin{bmatrix} \frac{1}{2}(h_t^2+e_x^2+e_y^2+e_z^2) & 0 & 0 & 0 \\ 0 & \frac{1}{2}(h_t^2+e_x^2-e_y^2-e_z^2) & -e_z h_t+e_x e_y & e_y h_t+e_x e_z \\ 0 & e_z h_t+e_x e_y & \frac{1}{2}(h_t^2-e_x^2+e_y^2-e_z^2) & -e_x h_t+e_y e_z \\ 0 & -e_y h_t+e_x e_z & e_x h_t+e_y e_z & \frac{1}{2}(h_t^2-e_x^2-e_y^2+e_z^2) \end{bmatrix}. \tag{3·14}$$

The condition for the equilibrium of the stress, of which Z_{22} to Z_{44} is the Euclidean representation, namely $\partial_i Z_{ik} = 0$ ($k = 2$ to 4) is, for $k = 2$,†

$$-\mathrm{i}\left\{\frac{\partial}{\partial x}\tfrac{1}{2}(h_t^2+e_x^2-e_y^2-e_z^2)+\frac{\partial}{\partial y}(e_z h_t+e_x e_y)+\frac{\partial}{\partial z}(-e_y h_t+e_x e_z)\right\} = 0. \tag{3·15}$$

When the charge is spherically symmetrical we must have h_t and e_r functions of the radial distance r only, and

$$e_x = \frac{x}{r}e_r, \quad e_y = \frac{y}{r}e_r, \quad e_z = \frac{z}{r}e_r. \tag{3·16}$$

Consequently (3·15) may be written

$$\frac{\partial}{\partial x}(\tfrac{1}{2}h_t^2)+\left\{\tfrac{1}{2}(x^2-y^2-z^2)\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}+xz\frac{\partial}{\partial z}\right\}\left(\frac{e_r^2}{r^2}\right)+3x\frac{e_r^2}{r^2}+\left(z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}\right)\left(\frac{e_r h_r}{r}\right) = 0. \tag{3·17}$$

The symmetry also gives $\quad \dfrac{\partial}{\partial x} = \dfrac{x}{r}\dfrac{\mathrm{d}}{\mathrm{d}r}, \quad \ldots, \quad \dfrac{\partial}{\partial z} = \dfrac{z}{r}\dfrac{\mathrm{d}}{\mathrm{d}r}.$

† *Note added by the author, June* 1958. For $\partial_k Z_{ik} = 0$ change signs of $e_z h_t$ and $-e_y h_t$. These cancel for symmetrical field.

and on using this the last expression in (3·17) becomes zero, and the equation reduces to

$$\frac{\mathrm{d}}{\mathrm{d}r}(\tfrac{1}{2}h_t^2+\tfrac{1}{2}e_r^2)+2\frac{e_r^2}{r}=0. \tag{3·18}$$

This is the differential relation between h_t and e_r required for the system to be in equilibrium. The same condition is obtained from the third and again from the fourth column.

Let Q_r be the total charge contained within a sphere of radius r, then

$$e_r = Q_r/4\pi r^2, \tag{3·19}$$

since any symmetrical charge outside the sphere will not affect e_r. Using this value and assuming that h_t and e_r are zero at $r=\infty$, we can integrate (3·18), obtaining

$$\tfrac{1}{2}h_t^2 = \int_r^\infty \frac{Q_r^2\,\mathrm{d}r}{8\pi^2r^5}-\frac{Q_r^2}{32\pi^2r^4} \tag{3·20}$$

for the h_t-energy density at a distance r from the centre. If there is no charge outside the r-sphere, Q_r is constant in the integral and the two terms cancel, making $\frac{1}{2}h_t^2$ zero; otherwise it has a positive value.

Another form of interest into which (3·18) may be put is

$$-\frac{\mathrm{d}}{\mathrm{d}r}(\tfrac{1}{2}h_t^2)=\frac{1}{(4\pi r^2)^2}Q_r\frac{\mathrm{d}Q_r}{\mathrm{d}r}, \tag{3·21}$$

since the other part of $\quad \frac{\mathrm{d}}{\mathrm{d}r}(\tfrac{1}{2}e_r^2), \quad$ viz. $\quad (\tfrac{1}{2}Q_r^2)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{(4\pi r^2)^2}\right),$

cancels with $2e_r^2/r$.

These equations, which express the condition that the stress is self-equilibrated (property (2)), are also consistent with the system being purely electromagnetic (property (1)). They satisfy the fundamental equations (I (4·11)) since the electric charge density

$$j_t = \mathrm{d}Q_r/(4\pi r^2\,\mathrm{d}r) \quad \text{is} \quad \operatorname{div}\mathbf{e} \tag{3·22}$$

with the magnetic charge density k_t, and the current densities, $\mathbf{j}$ and $\mathbf{k}$ put zero. For the magnetic current to be zero we must have

$$\mathbf{s}=\operatorname{grad}h_t=\frac{\mathrm{d}h_t}{\mathrm{d}\mathbf{r}}=-\frac{1}{h_t}\frac{Q_r}{4\pi r^2}\frac{\mathrm{d}Q_r/\mathrm{d}\mathbf{r}}{4\pi r^2}=-\frac{j_t}{h_t}\mathbf{e} \tag{3·23}$$

by (3·21). This is in accordance with the conservation conditions (3·4), since $j_t = r_t$.

To prove the property (3) assume a self-equilibrated system in which the total charge Q_a is symmetrically distributed inside a sphere of radius a, and consider the effect of adding to this a charge δQ_a contained in a shell of thickness δa at a. Let us associate with this shell a uniform energy density

$$\delta(\tfrac{1}{2}h_t^2)=\frac{Q_a\,\delta Q_a}{(4\pi a^2)^2} \tag{3·24a}$$

inside the sphere, dropping to zero in the shell and remaining zero outside, and an energy density, varying with the distance r from the centre, of

$$\delta(\tfrac{1}{2}e_r^2)=\frac{Q_a\,\delta Q_a}{(4\pi r^2)^2}, \tag{3·24b}$$

rising in the shell from zero to this value and extending outside it to infinity. If these densities are added to the $\frac{1}{2}h_t^2$ and $\frac{1}{2}e_r^2$ of the original system, then the original, the added, and the combined system each satisfies (3·21) and is in self-equilibrium. Thus (3·24) gives the overall increase of the energy density as we extend the equilibrated system by adding a shell of arbitrary charge. The resulting increase in the total h_t- and e_r-energy is

$$\delta w_0 = \int_0^a \delta(\tfrac{1}{2}h_t^2)\, 4\pi r^2\,\mathrm{d}r + \int_a^\infty \delta(\tfrac{1}{2}e_r^2)\, 4\pi r^2\,\mathrm{d}r = \frac{Q_a\delta Q_a}{12\pi a} + \frac{Q_a\delta Q_a}{4\pi a}. \tag{3·25}$$

As may be seen in (3·11) these values are those of a Lorentz shell system having an effective charge $q = \surd(2Q_a\delta Q_a)$. The result found there that after a classical transformation to motion the energy and momentum are exactly those of a classical particle (property (3)), i.e.

$$\delta w = \delta w_0/(1-v^2/c^2)^{\frac{1}{2}}, \quad \delta g = (v/c^2)\,\delta w,$$

must hold here also. By extending from $a = 0$ to $a = \infty$ the process of adding arbitrarily charged shells any spherically symmetrical distribution of charge can be built up; in it the energies δw_0 simply add together, and the property (3) continues to hold. The total energy w_0 of the equilibrated system, which is, on writing r for a in (3·25),

$$w_0 = \int_0^\infty \frac{Q_r(\mathrm{d}Q_r/\mathrm{d}r)\,\mathrm{d}r}{3\pi r}, \tag{3·26}$$

can be evaluated when Q_r is known as a function of r.

4. Transformation latitude

The discussion of rotation transformations in § 2 brought out the fact that the fundamental equations of the field theory remain invariant in form when all the complex vectors concerned, $(\psi + i\chi,\ e + ih,\ r + is)$, in the fourfold representation are given a common rotation, $\bar{f}$S, in which f is an arbitrary vector independent of x. This rotation may be applied at will to the field vectors without altering ∂ or the co-ordinate system in any way. There results a certain latitude in permissible transformations of the equations, which is a matter of some philosophic interest in electromagnetic theory.

When we think of an electromagnetic field as an explanation of some set of phenomena, we make a mental picture of a definite distribution of electric and magnetic forces, electric charge and current. It will be generally admitted, I think, that when we come down to fundamental ideas these quantities cannot be regarded as things directly observable at the various points of the field. The verification of the picture (when it is verified) comes rather from the calculated effects of the assumed field which do signify phenomena—effects reached by way of the mechanical properties, energy, momentum, and Lorentz force, attributed to the field, which can affect matter visibly. In standard theory once a particular field has been assumed and the co-ordinate system is not changed, there is only one such mental picture possible—the value of each variable is definitely fixed. It now appears that with the extended field equations there is an *infinite number of sets of values which may be assumed at will for these variables*, obtained by applying to the field the arbitrary locked rotation $\bar{f}$S, and each set is consistent with the phenomena to be explained. The fact must have some meaning in electromagnetic theory but it is not immediately obvious what this is. I found a study of its consequences on the system denoting a charged particle useful in discovering an answer.

(a) *Effect on a uniformly volume-charged electron*

The Lorentz electron with its charge confined to an infinitely thin surface shell has the defect of having infinite derivatives in the shell, and a better example for the purpose is that of a charge spread uniformly over the whole volume of a sphere, this being free of infinities. If Q is the total charge, ρ the charge density, and a the radius of the sphere, we easily get by (3·20)

$$h_t = (1/\sqrt{3})\,\rho(a^2-r^2)^{\frac{1}{2}}, \quad e_r = \tfrac{1}{3}\rho r,$$

inside the sphere, and outside

$$e_r = \tfrac{1}{3}\rho a^3/r^2. \tag{4·1}$$

With (inside)

$$r_t = j_t = \rho, \quad s_r = \frac{-\rho r}{\sqrt{3}\,(a^2-r^2)^{\frac{1}{2}}},$$

and all other variables zero these values give a self-equilibrated system.

(i) $\bar{f}$S *transformation*

To exhibit the effect of $\bar{f}$S on the $e+ih$ representative of this field let us simplify by writing, as in I (2·13),

$$f = |(f_t, \mathrm{i}f_x, 0, 0) = \left|\left(\frac{1}{(1-u^2/c^2)^{\frac{1}{2}}}, \frac{\mathrm{i}u/c}{(1-u^2/c^2)^{\frac{1}{2}}}, 0, 0\right),\right. \tag{4·2a}$$

where u is a velocity (approximately half the v of the Lorentz transformation), such that

$$u_{x,y,z} = \frac{c^2}{v^2}\left\{1-\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}\right\} v_{x,y,z} \tag{4·2b}$$

and

$$f_t^2+f_x^2 = \frac{1}{(1-v^2/c^2)^{\frac{1}{2}}} = \beta, \quad 2f_t f_x = \frac{v}{c}\beta = \alpha\beta. \tag{4·2b}$$

Inside the sphere the transformed field is readily found to be

$$h'_t = f_t h_t, \quad e'_r = f_t e_r, \tag{4·3a}$$

along with the following $\quad e'_t = -f_x e_x, \quad h'_h = -f_x h_t, \quad h'_\phi = f_x e_p.$

Outside we have only

$$e'_r = f_t e_r, \quad e'_t = -f_x e_x, \quad h'_\phi = f_x e_p. \tag{4·3b}$$

From these field values the total energy and x-momentum are easily calculated; they are found to be changed from $w_0 = Q^2/5\pi a$ and $g_x = 0$, respectively, to

$$w' = \beta w_0, \quad g'_x = \frac{\alpha\beta}{c} w_0 = \frac{v}{c^2} w' \tag{4·4}$$

with g'_y, g'_z remaining zero. Equation (4·4) gives just the *energy and momentum which the particle would have if it were moving along x with the velocity v*, but here, of course, there is no such motion—the co-ordinates have not been altered, and the sphere remains fixed in the space and unchanged in shape. The result draws our attention to the fact that it is not the energy and momentum, but the *resultant forward flux of energy* in the field, *that must be taken as the criterion of motion* in an electromagnetic system. Examination of this point by evaluating the total flux of energy through the plane $x = 0$ shows this to be zero, so that the flux is circuital only.

The new field like the old one satisfies the fundamental equations (I (4·10), (4·13)) when the $r_t \ldots s_z$ corresponding to $\bar{f}$S$(r+\mathrm{i}s)$ are introduced; in other words it is a purely electromagnetic self-equilibrated system. Alternatively, the classical form (I (4·11)) is satisfied when e'_t, h'_t, r, s are converted into the charge and current densities, j'_t, $\mathbf{j}$, k'_t, $\mathbf{k}$. Below are given the components of the latter which are not zero

$$j'_t = f_t\rho, \quad j'_x = \tfrac{2}{3}f_x\rho, \quad j'_\phi = -\frac{f_x\rho p}{\sqrt{3}\,(a^2-r^2)^{\frac{1}{2}}}, \quad k'_t = \frac{f_x\rho x}{\sqrt{3}\,(a^2-r^2)^{\frac{1}{2}}},$$

inside the sphere; and outside

$$j'_x = f_x\rho a^3(x^2-\tfrac{1}{3}r^2)/r^5, \quad j'_p = f_x\rho a^3 xp/r^5. \tag{4·5}$$

The total charge has been increased from $Q = \frac{4}{3}\pi a^3\rho$ to $f_t Q$. j'_x inside, and j'_x, j'_p, outside, express the density ($\times 1/c$) of a conserved current which flows through the sphere in the $+x$ direction, out at the front, and then symmetrically outwards curving round to enter the sphere again at the back. In addition there is a current given by j'_ϕ inside, circling negatively round the x-axis. Finally, there are magnetic charges (densities k'_t) developed inside the sphere, positive in front of and negative behind the centre.

(ii) $\bar{c}$R *transformation*

It is possible also to apply to the system (4·1) the non-classical rotations

$$e'+ih' = \bar{c}\mathrm{R}(e+ih), \quad r'+is' = \bar{c}\mathrm{R}^{-1}(r+is), \tag{4·6}$$

but in this case the co-ordinates must at the same time be given the inverse Lorentz transformation. This consequence is obtained from the general transformation formula (2·7) by putting $\bar{f}\mathrm{S} = \mathbf{1}$ instead of its classical value $\bar{c}$S. It seems like a simplification, but it is by no means so actually; the resulting field is as complicated as in the case just dealt with, rather more so indeed, because the sphere has contracted to a spheroid, and the variables are now functions of $x'-vt'$. Calculation of the total *energy* and *x-momentum* of the system shows that now these are *unchanged* from their values in the original stationary state. On the other hand, the total flux of energy through a fixed plane perpendicular to the motion has a finite value forwards; and the flux S' in fact satisfies the equation

$$\operatorname{div} \mathrm{S}' = v(\partial W'/\partial x'). \tag{4·7}$$

This is an equation the satisfying of which ensures that the flux gives just the same result for the energy density changes at a point as if the energy were being carried forward bodily with velocity v along x; for its two sides are the expressions—on the left by conservation principle, on the right by bodily motion theory—for the rate of decrease of the energy density at the point.

(iii) $\bar{c}$R $\bar{c}$S *transformation*

If a $\bar{c}$R-transformation is applied to the field ((4·2) to (4·5) with f written c) resulting from a $\bar{c}$S-transformation, the non-classical features the field possesses are annulled, and the net result is that of a classical transformation to motion with velocity v of the original static field. The new field values are

$$h'_t = \frac{1}{\sqrt{3}}\beta F'(a'^2-r'^2)^{\frac{1}{2}}, \quad e'_r = \tfrac{1}{3}\beta\rho r', \quad h'_\phi = \alpha\sin\theta' e'_r$$

inside the spheroid, and outside it

$$e'_r = \tfrac{1}{3}\beta\rho a'^3/r'^2, \quad h'_\phi = \alpha\sin\theta' e'_r. \tag{4·8}$$

The energy, momentum, and energy flux now behave all together in the standard way, the only new (and somewhat unexpected) feature being that there is now a circularly directed momentum density—$h'_t h'_\phi/c$ inside the spheroid (accompanied by an opposite flux of energy—a well-known effect when matter under tension moves, e.g. in the belt of a driven dynamo) which gives to the system a moment of momentum about OX of

$$-vQ^2/(64\sqrt{3}\,c^2).$$

A spherically symmetrical particle obviously cannot possess spin when it is at rest, but apparently here one develops along with motion. The only other variables remaining finite are the charge and current densities

$$j'_t = \beta\rho, \quad cj'_x = c\alpha\beta\rho \tag{4·9}$$

inside the spheroid. They satisfy the bodily transference condition

$$\operatorname{div} c\mathbf{j}' = v(\partial j'_t/\partial x'),$$

both inside, and at the surface of the spheroid.

(b) *Some general conclusions*

The example has shown that the electromagnetic field describing a stationary particle may be transformed at will into an equivalent field in which the energy and momentum (but not the energy flux) have the values that normally correspond to motion with an arbitrary velocity; and this may be done without the co-ordinate system being altered in any way, so that the particle continues definitely stationary.

(i) A little reflection shows that a similar transformation is possible in the ordinary theory of the mechanics of particles. When considering a system of colliding particles we normally reckon the energy and momentum of each to have its 'rest value' (m_0c^2 and 0, respectively) when the particle is at rest with respect to ourselves; but it is evident that no difference will be made to anything observable if we based the definition of the energies and momenta on some other assumed standard of rest. In the early classical days this idea was commonplace—the 'real' momentum of objects on earth was indeterminate, because that due to any assumed earth velocity (then merely to be added on) always cancelled in the calculation of observable effects in earth-bound systems. But it is equally a consequence of the relativity principle—although one tends to overlook it, for the resulting formulae are complicated and of no practical use. The matter becomes much simplified by dealing with the representation of the system in Euclidean 4-space. It may be summed up by saying that the energy-momentum vectors p of particles are not obliged to be parallel to the displacement vectors dx which mark their tracks in space and time. By assuming a suitable standard of rest for the definition of the energies and momenta the p's may be imagined lying at any given fixed angle to the dx's without this affecting any physical observable that comes in the representation, such as the calculated displacement vectors of visible particles after collisions. The p's thus admit an arbitrary Lorentz rotation to be applied to them all together independently of the co-ordinates; and this is exactly what has been found above to be the consequence of transformation latitude in the electromagnetic theory of particle constitution.

(ii) I have not seen the possibility of this latitude in the permissible transformations of either mechanical or electrical quantities referred to in standard works on the tensor

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calculus. (But here I should like to explain that throughout this work by 'standard' calculus I have meant that described by, for example, Eddington and Pauli in their books on relativity.) There a tensor is defined as a set of quantities which, on transformation of the co-ordinates, transforms in a single stated way for each contravariant, and each co-variant, aspect. For this to be the case the quantities must be regarded as having defined values for each set of co-ordinates. But for mechanical and electrical quantities there is a possibility of taking different definitions, and the quantities are not then fixed uniquely for each co-ordinate system until a choice has been made. In this event they may become open to transformations whose effect is virtually to change them from one definition to another without the co-ordinate system itself being changed. The 'latitude' transformation ($\mathcal{J}$S) is an example of such a transformation; perhaps it is not stretching the idea too much to regard the e-transformation by which in part I the matrix of mechanical components of matter, Z, is obtained from a matrix of the electrical components, as another, for it also is effected without change of the co-ordinate system—but it is a rather surprising change-over of definitions from a physical point of view. Evidently these are not tensor transformations as defined above, but so long as they do not change the tensor character of the quantities, they can easily be included in the scheme.

I should like here (for this seems the appropriate place) to make the following remarks in reference to a recent note by C. W. Kilmister (1953) on the subject of the earlier papers. Paper B was concerned with the result, quoted here in equation (2·1), concerning the relative transformational behaviour of a matrix and its e-transform. Inspection of these formulae makes it evident that, while either might be regarded as a standard tensor and submitted to a given transformation corresponding to that of the co-ordinates, the other with its different transformation cannot be a standard tensor (in the sense explained above) at the same time. Ultimately the reason is that the constant resultant magnitude which characterizes the relation between them is not an invariant in the standard tensor theory. Consequently it seemed useful to show that, by defining the covariant transformation rule in another than the standard way, a calculus could be devised in which the resultant magnitude *was* an invariant of the transformations, and in which a matrix and its e-transform could be regarded simultaneously as tensors of the same class. I do not think this calculus has much useful application in general relativity theory, but in the 'transformations at constant resultant magnitude' by which the relation between the field- and energy-stress matrices has been derived its use has been required in the present work. I regret, however, that through my ignorance of modern mathematical work I described it as an 'extension' of the tensor theory. Dr Kilmister, who in his note has extended and generalized some of my results, has pointed out that my extension is, or can be, included in the modern general theory of tensors, which has been considerably enlarged since Eddington's time by group theory and matric algebras. By his kindness I gather that the modern mathematical theory does not concern itself with the special characters of the covariant transformations and the corresponding invariants, but includes all the possible transformations together under group theory.

(iii) One has only to refer to the description below equation (2·5) to see that the electromagnetic picture resulting from the $\mathcal{J}$S transformation is much more complicated than the simple original one. It is understandable that when phenomena can be accounted for by different schemes we should make a point of choosing the simplest of these, and there is no question the standard theory gives this. But the fact that there are other possible ones throws light on the nature of some of the fundamental concepts concerned; and while it is comparatively easy to admit that the energy and momentum of a particle are not things having unique values with a given co-ordinate system, it is not so easy to accept charge, electric or magnetic, in the same category; yet these also have been changed by the transformation. The result is quite inconsistent with the older classic idea, on which the Lorentz restriction I (4·17) was founded, that electric charge is actually an objective entity, presented to us by Nature, which moves about bodily from place to place in the field.

The basis of this idea is, of course, the fact that as a result of the classic equations charge automatically satisfies an equation of continuity, $\bar{\mathrm{D}}\mathrm{j} = 0$. Yet the same result holds in the extended equations, as may readily be verified ($\bar{\mathrm{D}}\mathrm{k} = 0$ also). Nevertheless, the extended equations show (I (4·10 a)) that j, instead of being a single entity, is a composite structure, consisting of two parts, $\eta\mathrm{r}$ and $-\eta\mathrm{D}e_t$, and it is only to the sum of the two that the continuity equation applies. We cannot picture the parts by themselves as charges in motion; they must be regarded, like e and h, as quantities continuously varying with the time at each point of the field; yet their sum behaves as if it were something indestructible moving about in space, possessing an acquired individuality like a wave on a water surface. The Lorentz transformation alters the two parts in the same way, and consequently j behaves as a standard tensor; there is then nothing to come into conflict with the idea of the objectivity of charge. But the $\mathcal{J}$S-transformation alters the parts in different ways and hence j in a complicated way which *is* in conflict with that idea and this requires us, in fundamental theory at any rate, to recognize charge and current as essentially a mentally constructed entity.

5. Particles with integral numbers of units of charge and mass

In calculations on the construction of electromagnetic fields, which have the mechanical properties of particles, there are as many as four scalar variables at disposal—r_t, s_t (alternatively j_t, k_t), and e_t, h_t. It seems almost an embarrassment of riches for it leaves the charge, size, and mass of the particle correspondingly arbitrary. The number of variables might be reduced by imposing conditions on them.

In a spherically symmetrical static field the condition for self-equilibrium (3·23) takes the form (writing ρ for j_t, e_r for $\mathbf{e}$)

$$\frac{\mathrm{d}h_t}{\mathrm{d}r} = -\frac{\rho}{h_t}e_r,$$

so that if we put

$$\rho = \kappa h_t, \tag{5·1}$$

then

$$\frac{\mathrm{d}h_t}{\mathrm{d}r} = -\kappa e_r. \tag{5·2}$$

κ is not a constant—for example, in the uniformly volume-charged sphere by (4·1) it varies from $\sqrt{3}/a$ at $r = 0$ to ∞ at $r = a$. The condition that κ should be constant over the whole sphere seemed to have possibilities and this case is investigated below. We shall then have for (5·2), since

$$e_r = \frac{Q}{4\pi r^2} \quad \text{and} \quad Q = \int_0^r 4\pi r^2 \rho \,\mathrm{d}r, \tag{5·3 a, b}$$

$$\frac{\mathrm{d}h_t}{\mathrm{d}r} = -\frac{\kappa_2}{r^2}\int_0^r r^2 h_t \mathrm{d}r.$$

This gives the second-order equation

$$\frac{d^2h_t}{dr^2}+\frac{2}{r}\frac{dh_t}{dr}+\kappa^2h_t=0 \tag{5·4a}$$

the solution of which is $h_t=(A\sin\kappa r+B\cos\kappa r)/r$,

where A and B are constants. On assuming that h_t is not infinite at $r=0$, the solution reduces to

$$h_t=(A\sin\kappa r)/r. \tag{5·4b}$$

Starting with the value κA at the centre, h_t is reduced to zero at a distance $r=\pi/\kappa$. From this distance on we can complete the scheme by the plain electrostatic field of the total charge, and the whole system will possess the electrical and mechanical properties proved in § 3. But (5·4b) also draws attention to an idea not there considered that, without breaking the condition for equilibrium (which depends only on the squares of h_t and e_r), h_t might be positive in some places and negative in others. It indicates the mathematical possibility of alternating shells of positive and negative h_t, extending with a gradual diminution of amplitude to infinity. The alternating system can also be terminated at any point where $h_t=0$, if continued by the electrostatic field of the contained charge. Since $\rho=\kappa h_t$, the solution will also involve the presence in alternate shells of positive and negative charges. On classical lines it is of course impossible to picture such charges freely suspended in space, but with the extended theory that difficulty no longer exists.

The charge contained in a sphere of radius r is by (5·3b), (5·1) and (5·4b)

$$Q=4\pi\kappa A\int_0^r r\sin\kappa r\,dr$$
$$=4\pi A\left(-r\cos\kappa r+\frac{\sin\kappa r}{\kappa}\right). \tag{5·5}$$

At the radii at which h_t is zero r is $n\pi/\kappa$, and

$$Q=\pm4\pi^2An/\kappa, \tag{5·6}$$

according as n is odd or even. The charge in successive shells, alternately + and −, goes on increasing without limit. There are radii at which the total charge is zero, but they are not points at which the shell system can be terminated, since h_t is not zero at them, and a Lorentz shell of infinite charge density (not only a somewhat artificial concept, but one which brings back the charge) would have to be assumed to effect an equilibrated change to an electrostatic field.

By (5·4b), (5·5), and (5·3a) the energy density at r is

$$\tfrac{1}{2}(h_t^2+e_r^2)=\frac{1}{2}\frac{A^2}{r^2}\left(1-\frac{2\cos\kappa r\sin\kappa r}{\kappa r}+\frac{\sin^2\kappa r}{\kappa^2r^2}\right).$$

Multiplying this by $4\pi r^2dr$ and integrating from 0 to r, one gets for the energy inside a sphere of radius r the value

$$2\pi A^2r\left(1-\frac{\sin^2\kappa r}{\kappa^2r^2}\right), \tag{5·7}$$

which reduces to $2\pi^2A^2n/\kappa$ for $r = n\pi/\kappa$. On continuing from here by the electrostatic field necessary for equilibrium, namely $e_r = \pm 4\pi^2An/(\kappa\, 4\pi r^2)$, the energy external to the sphere is

$$\frac{1}{2}\int_{n\pi/k}^{\infty} e_r^2 4\pi r^2\,\mathrm{d}r = 2\pi^2A^2n/\kappa. \tag{5·8}$$

Whatever the size of the sphere, there is the same amount of energy inside and outside it, and the total rest energy of the system is

$$w_0 = 4\pi^2A^2n/\kappa = AQ_0, \tag{5·9}$$

where Q_0 is the magnitude of the total charge Q, whether this is positive or negative (n odd or even).

The system which results from the simple condition that ρ/h_t shall be constant is of unexpected interest. It is not surprising that the condition should fix the size of the particle, but there is the additional result that the solution gives a series of particles, which are only capable of existing when their charges and energies are integral multiples of the smallest values. The mathematical reason for this is that the equation for h_t (5·4a) is a form of wave equation, and the shells bounded by zero h_t are successive half-waves of a spherical train of constant wavelength $2\pi/\kappa$. The waves are very different from any contemplated in classical electromagnetic theory. The field changes along them, not from a transverse electric to a perpendicular magnetic force, but from a scalar h_t to a radial electric force, these quantities like the former having distinguishable kinds of energy.

While still recognizing the ultimate inadequacy of the classical description of particles, one may regard this system of shells as a very simplified classical model of a compound-charged particle in which each constituent particle has gone into the form of a concentric wave shell. This is feasible because each full wavelength shell, i.e. from $r = n\pi/\kappa$ to $r = (n+2)\,\pi/\kappa$ (n odd), carries the same charge

$$q = 8\pi^2An/\kappa, \tag{5·10}$$

and an energy in proportion. (A half-wavelength has a charge greater than the whole system but without the corresponding energy, and it could not exist as a particle.) The system is rather like our idea of an atomic nucleus, although perhaps it is too simple and idealized to be of practical use in that theory. Nevertheless, it seems distinctly of interest that the shell constitution should have been presented so readily by the extended electromagnetic theory, and in the next section it is shown that the most straightforward simplification possible of the general equations in fields containing charge, leads to the wave equation which produced these results.

6. Relation of field equations to wave theory

In part I the field equations were obtained in the form of a differential relation (I (4·9))

$$\bar{\partial}\mathrm{R}^{-1}(e+\mathrm{i}h) = r+\mathrm{i}s, \tag{6·1}$$

defining the derived field $r+\mathrm{i}s$ of an arbitrary vector field $e+\mathrm{i}h$, the two fields being subject to the 'conservation restriction' (I (4·13b) also (2·12a)), which will here be written

$$\bar{r}e = -\bar{s}h, \quad \bar{r}\mathrm{S}_k^{-1}h = \bar{s}\mathrm{S}_k^{-1}e \quad (k = 2 \text{ to } 4). \tag{6·2}$$

The problem suggested by the last section is to combine (6·1) and (6·2) into a single linear form with the least possible reduction in generality.

In (6·2) e, h, r, s are not vectors in the sense of the standard calculus; each is a set of four all-real components in a manifold in which x_1 is real and x_2 to x_4 are imaginary; nevertheless, for the present purpose we can imagine them as represented by lines in the Euclidean 4-space. Equation (6·2) then indicates that, apart from a change of sign, r stands in the same relation to e and h, as s does to h and e. Let us consider the restriction that the four scalar products $\bar{r}e$ and $\bar{r}S_k^{-1}h$ $(k = 2$ to $4)$ are each zero. This means that r is to be perpendicular to e and also to the three lines $S_2^{-1}h$, $S_3^{-1}h$, $S_4^{-1}h$. These three lines are mutually perpendicular, and each is perpendicular to h. Hence r is parallel to h, and (6·2) now requires in a similar way that s is perpendicular to h and parallel to e. It follows also that e must be perpendicular to h, or

$$\bar{e}h = 0, \quad \text{with also} \quad \bar{r}s = 0. \tag{6·3}$$

The restriction thus makes the field completely orthogonal, e and s alined together being perpendicular to h and r alined together.

The simplified class of field thus arrived at is characterized by a corresponding simplification in the value of its resultant magnitude. By I (3·14) this is

$$Z_3 = \{(\bar{e}e - \bar{h}h)^2 + 4(\bar{e}h)^2\}^{\frac{1}{2}}, \tag{6·4}$$

and when $\bar{e}h = 0$, Z_3 undergoes the considerable reduction to

$$Z_3 = \pm(\bar{e}e - \bar{h}h). \tag{6·5}$$

Suppose that this field at any point is rotated in the fourfold until it reaches the original rest-state contemplated in I (3·1), where it will be represented by two lines, $e^0 + \mathrm{i}h^0$, $e^0 - \mathrm{i}h^0$, of magnitudes $e_0 + \mathrm{i}h_0$, $e_0 - \mathrm{i}h_0$, each parallel to the axis 1. In the general case the resultant magnitude (6·4) is reduced to

$$Z_3 = \{(e_0^2 - h_0^2)^2 + 4(e_0 h_0)^2\}^{\frac{1}{2}} = \pm(e_0^2 + h_0^2), \tag{6·6}$$

while (6·5) reduces to

$$Z_3 = \pm(e_0^2 - h_0^2), \tag{6·7}$$

and these cannot agree unless either e^0 or h^0 is zero. The limitation to orthogonal fields is thus equivalent to the assumption that the complexity of the vectors can be transformed away (or regarded as produced) by real rotations. This seems a reasonable assumption to make, so long as it is a single primitive field such as might constitute a particle that is considered (for orthogonality is not invariant to superposition of fields). In I (3·1) the rest-state field is regarded as the result of factorizing the rest density of fundamental matter, and there seems no special reason for making the factors complex.

The field being assumed orthogonal, r and s can be eliminated from (6·1) by writing $r = \kappa h$, $s = \kappa' e$, but the resulting equation will not be invariant in form when two such fields are superposed unless $\kappa' = \kappa$ and κ is the same for both fields. Accepting this we get for the equation

$$\partial \mathrm{R}^{-1}(e + \mathrm{i}h) = \kappa(h + \mathrm{i}e) = \mathrm{i}\kappa(e - \mathrm{i}h). \tag{6·8}$$

The condition for κ requires it to be either a function of the absolute position in the fourfold, which would be meaningless, or a constant; so we are practically committed to $\kappa =$ constant as a further restriction to be imposed. κ has the dimensions of (length)$^{-1}$, and if its value were known this would fix the linear size of an electron constructed in accordance with (6·8), but there is nothing in this analysis to determine a numerical value for it.

One has now to ask if (6·8) is a satisfactory equation in that it keeps its form invariant under a Lorentz transformation of co-ordinates. For this to be the case the first condition evidently is that κ must be invariant. Secondly, the equation has to satisfy the general transformation conditions of (6·1). These have been obtained in (2·7), and they show that when, after transformation, $e+\mathrm{i}h$ becomes $\bar{c}\mathrm{R}\bar{f}\mathrm{S}(e+\mathrm{i}h)$, the right-hand side of (6·8), which stands for and must transform in the same way as $r+\mathrm{i}s$, will require that

$$(e-\mathrm{i}h)' = \bar{c}\mathrm{R}^{-1}\bar{f}\mathrm{S}(e-\mathrm{i}h).$$

But as the conjugate of $(e+\mathrm{i}h)$ this must also take the form

$$(e-\mathrm{i}h)' = \bar{c}^*\mathrm{R}\overline{f^*}\mathrm{S}(e-\mathrm{i}h).$$

The condition for invariance of form thus is

$$\bar{c}^*\mathrm{R} = \bar{c}\mathrm{R}^{-1}, \quad \overline{f^*}\mathrm{S} = \bar{f}\mathrm{S}. \tag{6·9}$$

It will be satisfied if $\bar{c}\mathrm{R}, \bar{f}\mathrm{S}$ denote rotation through an imaginary and a real angle, respectively. The former is the more usual type of rotation, but the latter can always be applied, since it is arbitrary, the simplest case being got by taking $\bar{f}\mathrm{S}$ as **1**. But it must be noted that (6·8) is not invariant to the more limited transformation of the classical field equations, for this requires f to be identified with c, and it is then impossible to satisfy the second condition of (6·9).

When κ is real, (6·9) is a first-order form of a second-order wave equation, which is readily obtained from it as follows. In order to show the i's in them explicitly write

$$\bar{\partial}\mathrm{R} = \partial_{ct}-\mathrm{i}\partial_k\mathrm{R}_k, \quad \bar{\partial}\mathrm{R}^{-1} = \partial_{ct}+\mathrm{i}\partial_k\mathrm{R}_k \quad (k = x, y, z),$$

Then (6·9) gives

$$\bar{\partial}\mathrm{R}^{-1}(e+\mathrm{i}h) = (\partial_{ct}e-\partial_k\mathrm{R}_kh)+\mathrm{i}(\partial_{ct}h+\partial_k\mathrm{R}_ke)$$
$$= \kappa h+\mathrm{i}(\kappa e);$$

and

$$\bar{\partial}\mathrm{R}\,.\,\bar{\partial}\mathrm{R}^{-1}(e+\mathrm{i}h) = \kappa(\partial_{ct}-\mathrm{i}\,\partial_k\mathrm{R}_k)\,(h+\mathrm{i}e)$$
$$= \kappa(\partial_{ct}h+\partial_k\mathrm{R}_ke)+\mathrm{i}(\partial_{ct}e-\partial_k\mathrm{R}_kh),$$

or

$$\bar{\partial}\partial(e+\mathrm{i}h) = \kappa^2(e+\mathrm{i}h). \tag{6·10}$$

This is a well-known form of wave equation in which the phase velocity of a train of plane waves is different from c and moreover depends on the wavelength, so that the train is subject to dispersion. It is, I think, a new result to find that the electromagnetic equations (as extended here, but still in their fundamental form and not modified by introducing factors like μ and K to make them apply to material media) can include the wave form (6·10). This equation bears an obvious resemblance to the form

$$\bar{\partial}\partial\psi = -\kappa^2\psi, \tag{6·11}$$

which is the basis of the wave theory of particles. At first sight it looks as though the two equations might easily be identified by assuming the κ in (6·10) to be imaginary, but a repetition of the deduction with κ put i × (real number) shows that the suggestion fails; the identifications now have to be made differently, and the final result is unchanged from (6·10).

The general case in which κ is complex may be discussed by considering the result for the vector potential field ϕ of the electromagnetic identity (I (4·12)). Suppose ϕ to be a field limited to satisfying the equation

$$\bar{\partial}\mathrm{R}\phi = \kappa\phi^*. \tag{6·12}$$

Then, since the conjugate equation must also hold, and all the R_α are real,

$$\overline{\partial^*}\mathrm{R}\phi^* = \kappa^*\phi.$$

Consequently, $$\overline{\partial^*}\mathrm{R}\,.\,\bar{\partial}\mathrm{R}\phi = \kappa\overline{\partial^*}\mathrm{R}\phi^* = \kappa\kappa^*\phi,$$

and this is the same as $$\bar{\partial}\partial\phi = \kappa\kappa^*\phi. \tag{6·13}$$

Thus whether κ is real or imaginary the coefficient of ϕ in (6·13) must be a positive real. The use of a quaternionic set (R or another) is essential for obtaining the electromagnetic equations, and it cannot be applied to (6·11).

As is well-known, Dirac discovered a method of factorizing $\bar{\partial}\partial$ in (6·11) so as to get a first-order equation, which is

$$\partial_\alpha \mathrm{E}_\alpha \psi = \kappa\psi \tag{6·14}$$

in E-number notation. If the $\mathrm{E}_1 \,..\, \mathrm{E}_4$ are mutually anticommuting 4^2 matrices, each a square root of -1, we have

$$\partial_\beta \mathrm{E}_\beta\,.\,\partial_\alpha \mathrm{E}_\alpha \psi = -(\partial_1^2 + \ldots + \partial_4^2)\,\psi = \kappa^2\psi,$$

and (6·11) is always satisfied. (6·14) is somewhat similar to the electromagnetic (6·12) but the E_α are not quaternionic (in a simple case we might put $\mathrm{E}_1 \,..\, \mathrm{E}_4 = \mathrm{R}_2, \mathrm{R}_3, \mathrm{iR}_4\mathrm{S}_2, \mathrm{iR}_4\mathrm{S}_3$), and so the equations are intrinsically different.

In the two second-order equations,

$$\bar{\partial}\partial\phi = \kappa^2\phi \quad \text{and} \quad \bar{\partial}\partial\psi = -\kappa^2\psi,$$

while there is an evident correspondence which may be significant, the fundamental character of the difference between them is demonstrated by the fact that the former requires the wave velocity to be less than, and the latter greater than, c. The plane wave form $A \sin 2\pi(\nu t - kx)$ satisfies the ϕ-equation only if the phase velocity ν/k is $c(1-\kappa^2/4\pi^2k^2)$, and the ψ-equation only if $\nu/k = c(1+\kappa^2/4\pi^2k^2)$. The ϕ-equation denotes the existence of a special kind of electromagnetic waves, the velocity of which, as might be expected, is always less than that of light, in a certain class of fields recognized as possible in the extended theory. The waves of the ψ-equation, on the other hand, are the de Broglie waves of velocities greater than light which are associated in modern theory with the energies and momenta of moving particles. In the common use of this equation a particle is treated as an object whose position and momentum, time and energy, are not open to exact knowledge, and have to be stated in terms of a probability determined from the wave field of ψ. The mathematical reason why the $-$ sign comes into the ψ-equation is that the differential operator which represents the *classic* energy momentum (i.e. not the quantum value $h\nu$, hk) is taken to be an imaginary quantity of the form $\mathrm{i}\partial$. Justifiable as this is for its purpose, it is outside the scope of the ϕ-equation, or the electromagnetic equations, which are built on the view that the classic energy momentum is represented by a real number. In their application to particles these appear to serve a different purpose from the ψ-equation—although one has to recognize that the latter is used extensively in modern work as the basis of theories of the constitution of nuclear particles (Peierls 1953). In the present paper, in seeking to develop the theory of particle constitution, the aim has been to pursue the classical theory on its own lines until the quantum considerations of probabilities are required. The construction of the elementary model atom in the previous section is, in fact, an example of the application of the ϕ-equation in the form (6·10) to a spherically symmetrical system.

References

Kilmister, C. W. 1953 *Proc. Roy. Soc.* A, **218**, 144.
Laporte, O. & Uhlenbeck, G. E. 1931 *Phys. Rev.* **37**, 1380.
Lorentz, H. A. 1909 *The theory of electrons*. Leipzig : B. G. Teubner.
Milner, S. R. 1952 *Proc. Roy. Soc.* A, **214**, 292, 312, referred to as 'A' and 'B'.
Peierls, R. E. 1953 *Proc. Phys. Soc.* A, **66**, 313.

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